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Mean Values of L-Funtions Associated to Elliptic, Fermat and Other Curves at the Centre of the Critical Strip*

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1. NOTATION

Let $(a(n))$ be a sequence of complex numbers such that the Dirichlet series

$$\sum_{n=1}^{\infty} a(n)\, n^{-s}$$

converges absolutely in some half-plane $\operatorname{Re} s > \sigma_0$. We define

$$L_i(s) = \sum_{n=1}^{\infty} a(n^i)\, n^{-s} \qquad (i = 1, 2, \ldots).$$

For a Dirichlet character χ we also define the twisted series

$$L_i(s, \chi) = \sum_{n=1}^{\infty} a(n^i)\, \chi(n)\, n^{-s}.$$

We shall be concerned with real primitive Dirichlet characters $\chi \bmod |d|$, i.e.,

$$\chi(n) = \left(\frac{d}{n}\right),$$

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the Kronecker symbol where d is the discriminant of a quadratic field. Such discriminants will be called fundamental in the sequel.

Let N and R be fixed integers. For any character χ mod $|d|$ as above with $(d, NR) = 1$, we assume that $L_1(s, \chi)$ satisfies a functional equation of the following type:

$$A_\chi{}^s T_\chi(s) L_1(s, \chi) = w_\chi A_\chi^{k-s} T_\chi(k - s) L_1(k - s, \chi), \tag{1.1}$$

where

$$A_\chi > 0, \qquad k > 0,$$
$$w_\chi = w\epsilon(d)\,\chi(R)$$

with $|w| = 1$ and ϵ a primitive Dirichlet character mod N, and where $A_\chi{}^s T_\chi(s)\, L_1(s, \chi)$ is an entire function of s. Here $T_\chi(s)$ denotes a product of gamma-factors

$$T_\chi(s) = \begin{cases} \displaystyle\prod_{i=1}^{J^+} \Gamma(s + \alpha_i{}^+) & \text{if } \ d > 0 \\[2ex] \displaystyle\prod_{i=1}^{J^-} \Gamma(s + \alpha_i{}^-) & \text{if } \ d < 0 \end{cases}$$

for positive integers J^+, J^- and real numbers $\alpha_i{}^+, \alpha_i{}^- > -k/2$ depending only on $L_1(s)$. By abuse of notation we shall write

$$T_\chi(s) = \prod_{i=1}^{J} \Gamma(s + \alpha_i), \tag{1.2}$$

it being clear that J and α_i depend on the sign of d. The restrictive assumptions on the form of the gamma-factors have been made in order to considerably simplify the proofs. While they do cover most of the important cases, it is clear that more relaxed hypotheses could be assumed.

As a further condition, we let

$$A_\chi = f(|d|), \tag{1.3}$$

where $f(x)$ is a non-decreasing C^1 function of $x \geqslant 1$.

2. Introduction

The assumptions of Section 1 are quite natural and suggest that $L_1(s)$ may be associated to a modular form. In fact Weil [13] has shown that if $L_1(s)$ satisfies

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)\, L_1(s) = w_1 \left(\frac{\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)\, L_1(k-s)$$

where N is a positive integer, and in addition

$$\left(\frac{\sqrt{N}}{2\pi} q\right)^s \Gamma(s)\, L_1(s,\chi) = w_\chi \left(\frac{\sqrt{N}}{2\pi} q\right)^{k-s} \Gamma(k-s)\, L_1(k-s,\bar{\chi})$$

for "sufficiently many" q and all primitive characters $\chi \bmod q$, where $w_\chi = w_1\epsilon(q)(\tau_\chi/\tau_{\bar{\chi}})\,\chi(-N)$, $\tau_\chi = \sum_{a=1}^{q} \chi(a)\, e^{2\pi i(a/q)}$ is the Gauss sum and ϵ is a primitive Dirichlet character mod N, then

$$\varphi(z) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z}$$

is a modular form with multiplier ϵ of weight k for the congruence subgroup

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbf{Z},\, ad - bc = 1,\, c \equiv 0 \ (\mathrm{mod}\ N)\right\}.$$

If in addition $L_1(s)$ is an L-function associated to an abelian variety $\mathscr{A}$ of dimension δ over $\mathbf{Q}$, the general conjectures of Birch and Swinnerton–Dyer give a meaning to the value of $L_1(s)$ at $s = k/2$ (see [9]). Here $L_1(s)$ is defined by an Euler product

$$L_1(s) = \prod_p \prod_{i=1}^{2\delta} (1 - \beta_{p,i}\, p^{-s})^{-1}$$

where (except for a finite number of bad primes) $|\beta_{p,i}| = p^{1/2}$ and $\prod_{i=1}^{2\delta} (1 - \beta_{p,i})$ is the number of rational points on $\mathscr{A}$ (mod p).

For example if $\mathscr{A}$ is an elliptic curve over $\mathbf{Q}$, Weil [13] conjectured that the L-function $L_1(s)$ associated to $\mathscr{A}$ comes from a modular form of weight $k = 2$ for the congruence subgroup $\Gamma_0(N)$, where N is the conductor of $\mathscr{A}$. In this case, the conjecture of Birch and Swinnerton–Dyer says that $L_1(s)$ has a zero at $s = 1$ of order r, the rank of the Mordell–Weil group $\mathscr{A}(\mathbf{Q})$; and that

$$\lim_{s\to 1} \frac{L_1(s)}{(s-1)^r} = \alpha[\text{Ш}]\, \frac{\det\langle P_i, P_j\rangle}{(\mathscr{A}(\mathbf{Q})\colon \mathbf{B})^2} \prod_{p|\Delta} c_p\,,$$

where $P_1, \ldots, P_r$ are r independent points in $\mathscr{A}(\mathbf{Q})$, $\mathbf{B} = \sum \mathbf{Z}P_i$, [Ш] is the order of the Tate–Šafarevič group, $\langle\ ,\ \rangle$ is the height pairing, etc. (see [10] for the definitions of all the symbols occurring above).

Another example is given by the Jacobian variety $\mathscr{J}$ of the Fermat curve

$$\mathscr{F}\colon x^n + y^n = 1.$$

It is known (see [12]) that $L_1(s)$, the L-function of $\mathscr{F}$, is a Hecke L-series with Grössencharakter of the cyclotomic field $\mathbf{Q}(\sqrt[4]{1})$ and, therefore, satisfies the assumptions of Section 1. Note, however, that in this case $T_\chi(s)$ will have several gamma-factors ([4]). According to the conjecture of Birch and Swinnerton–Dyer, $L_1(1)$ should vanish if $\mathscr{J}$ has infinitely many rational points.

The conjecture of Birch and Swinnerton–Dyer for L-functions associated to abelian varieties can be considered as a geometric analogue of Dirichlet's class number formulae. In Gauss' Disquisitiones [3, Section 302] one already finds conjectures about the average distribution of class numbers, and this topic has been subsequently dealt with by several authors [1, 7, 11]. It has occurred to us that these results might be capable of considerable generalization.

If $L_1(s)$ is the L-function of an Abelian variety $\mathscr{A}$ over $\mathbf{Q}$, and for a fundamental discriminant d

$$\chi(n) = \left(\frac{d}{n}\right) \qquad \text{(Kronecker's symbol)},$$

then

$$L_1(s)\, L_1(s, \chi) \tag{2.1}$$

is the L-function of $\mathscr{A}$ over $\mathbf{Q}(\sqrt{d})$. We are, therefore, led to study the distribution of the values of (2.1) at $s = k/2$ (note that Weil's general conjecture predicts that $L_1(s)$ comes from a modular form of weight $k = 2$). We will show that the mean value

$$\sum_{|d|\leqslant D}' L_1\left(\frac{k}{2}, \chi\right),$$

where the prime indicates that d runs over fundamental discriminants such that $(d, NR) = 1$, is strictly related to the Mellin transform of a function involving $L_2(s)$. It is interesting to remark that in many cases $L_2(s)$ can be interpreted in terms of the L-function of $\mathscr{A} \times \mathscr{A}$ [8]. Our results can be put in a precise form as follows.

THEOREM (1). *Let $L_1(s)$ satisfy the assumptions of Section 1. Then for $D \to \infty$, B, c large constants, $\epsilon > 0$ arbitrarily small, and $F = Bf(D) \times \log^J(1 + f(D))$, we have*

$$\sum_{|d|\leqslant D}' L_1\left(\frac{k}{2}, \chi\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{|d|\leqslant D}' (1 + w_\chi)\frac{T_\chi(k/2 + z)}{T_\chi(k/2)}\, L_2(k + 2z, \psi)\, f(|\,d\,|)^z \frac{dz}{z} + O\left(D^{1/2}(\log D)\left(1 + \sum_{n\leqslant F} |\,a(n)|\, n^{-k/2+3/16+\epsilon}\right)\right)$$

where $\chi(n) = (d/n)$, ψ *is the principal character* mod $|d|$, *and the sums* $\sum'$ *run over fundamental discriminants* d *such that* $(d, NR) = 1$.

In the case that $k \geqslant 1$, if we further assume a Ramanujan–Petersson-type bound

$$|a(n)| \ll n^{(k-1)/2+\epsilon}, \tag{2.2}$$

it immediately follows that

$$\sum_{n \leqslant F} |a(n)| n^{-k/2+3/16+\epsilon} \ll f(D)^{11/16+\epsilon}.$$

Furthermore, since $L_2(s, \psi)$ converges absolutely for $\operatorname{Re} s > k$, by (2.2), the line of integration in Theorem (1) can be moved to ϵ. Hence we obtain the following

COROLLARY. *Let* $L_1(s)$ *satisfy the assumptions of Section* 1 *and* (2.2). *Then, as* $D \to \infty$,

$$\sum_{|d| \leqslant D}' L_1\left(\frac{k}{2}, \chi\right) \ll \{D + D^{1/2}(\log D) f(D)^{11/16}\} f(D)^{\epsilon}.$$

The exponent 3/16 occurring in the error term of Theorem (1) arises from Burgess' estimate

$$\sum_{m=h+1}^{h+t} \chi(m) \ll t^{1/2} q^{3/16+\epsilon} \tag{2.3}$$

for any nonprincipal Dirichlet character χ mod q (Theorem (2), with the choice $r = 2$, in [2]). We remark that if one had the expected hypothetical estimate

$$\sum_{h+1}^{h+t} \chi(m) \ll t^{1/2} q^{\epsilon}$$

in place of (2.3), then the above corollary would be improved to

$$\sum_{|d| \leqslant D}' L_1\left(\frac{k}{2}, \chi\right) \ll \{D + (Df(D))^{1/2}(\log D)\} f(D)^{\epsilon}.$$

The integral in Theorem (1) can be evaluated by shifting the line of integration and applying the residue theorem. If $L_2(s)$ has an Euler product

$$L_2(s) = \prod_p \prod_{i=1}^{2\lambda} (1 - \gamma_{p,i} p^{-s})^{-\delta_i}$$

and a pole of order $\rho \geqslant 0$ at $s = k$, we expect that

$$\sum_{|d| \leqslant D}' L_1\left(\frac{k}{2}, \chi\right)$$

$$\sim \frac{1}{\rho!} [z^\rho L_2(k + 2z)]_{z=0} \sum_{|d| \leqslant D}' (1 + w_\chi) \log^\rho f(|d|) \prod_{p|d} \prod_{i=1}^{2\lambda} (1 - \gamma_{p,i} p^{-k})^{\delta_i}$$

although we have been unable to prove this.

Let $f(z)$ be a cusp form of weight 2 for the congruence subgroup $\Gamma_0(N)$. If E is the associated elliptic curve of conductor N then the Hasse–Weil L-function is

$$L_E(s) = \sum_{n=1}^{\infty} a(n) n^{-s} = \prod_{p|N} \left(1 - \frac{a(p)}{p^s}\right)^{-1} \prod_{p \nmid N} \left(1 - \frac{\gamma_p}{p^s}\right)^{-1} \left(1 - \frac{\bar{\gamma}_p}{p^s}\right)^{-1}$$

where $a(p) = \pm 1$ or 0 if $p \mid N$ and γ_p is the "Trace of Frobenius" which satisfies $|\gamma_p| = p^{1/2}$. Putting

$$\alpha = \lim_{s \to 2} (s - 2) \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}$$

the above conjecture asserts

$$\sum_{|d| \leqslant D}' L_E(1, \chi) \sim \alpha \prod_{p|N} \left(1 - \frac{1}{p}\right)^{-1} \sum_{|d| \leqslant D}' (1 + (-1)^r \chi(-N))$$
$$\times \prod_{p|d} \left(1 - \frac{\gamma_{p^2}}{p^2}\right)\left(1 - \frac{\bar{\gamma}_{p^2}}{p^2}\right)\left(1 + \frac{1}{p}\right)^{-1}$$

where r is the rank of the Mordell–Weil group of E over $\mathbf{Q}$. Note that

$$\alpha = \frac{96\pi}{N} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \langle f, f \rangle$$

where $\langle f, f \rangle$ is the Petersson inner product of f with itself. It follows that α can always be expressed in terms of the periods of E.

In the special case when $N = 11$, we have

$$\alpha = \frac{2}{\pi} \Omega^+ \Omega^-$$

where $\Omega^+ = 0.6346047$ is the real period and $\Omega^- = 1.4588166$ is the absolute value of the imaginary period. Our conjectures assert that the average value of $L_E(1, \chi)$, for $\chi(\mathrm{mod}\, p)$ with p a rational prime satisfying $\chi(-11) = 1$, should be

$$\text{average value of } L_E(1, \chi) \sim \frac{22}{5\pi} \Omega^+\Omega^-.$$

Let $\tau(\chi)$ be the Gauss sum and define

$$Ш_p = \begin{cases} \tau(\chi)\, L_E(1, \chi)/2\Omega^-, & \chi(-1) = -1, \\ \tau(\chi)\, L_E(1, \chi)/10\Omega^+, & \chi(-1) = +1, \end{cases}$$

so that according to the conjecture of Birch and Swinnerton–Dyer, $Ш_p$ coincides (for $L_E(1, \chi) \neq 0$) with the order of the Tate–Šafarevič group of E over $Q(\sqrt{\chi(-11)p})$. It follows that

$$\text{average value of } Ш_p \sim \begin{cases} (11\Omega^+/5\pi)\sqrt{p}, & \chi(-1) = -1, \\ (11\Omega^-/25\pi)\sqrt{p}, & \chi(-1) = +1. \end{cases}$$

This is the elliptic analogue of Gauss' conjecture (see Disquisitiones [3, Section 302])

$$\text{average value of } h(d) \sim \frac{2\pi}{7\zeta(3)} \sqrt{d}$$

where $h(d)$ is the number of classes of binary quadratic forms with discriminant $-d$.

For the elliptic curve of conductor $N = 11$, Glenn Stevens of Harvard University has provided numerical evidence supporting the above conjectures. Define $A(Q)$ to be the average value of $L_E(1, \chi)$ for prime conductors $p \equiv 1(4)$, $p \leqslant Q$, satisfying $\chi(-11) = +1$. Extracting from Stevens' computer printout

Q	$A(Q)$
37	1.130985
389	1.282885
683	1.219574
1021	1.181417
1409	1.349425
1709	1.273069
2039	1.329763
2357	1.325873
2699	1.322410
2927	1.285612

The values of $A(Q)$ are converging rather well to the conjectured average value of 1.2966... .

Specializing to the case $L_1(s, \chi) = L(2s, \chi)$, the classical Dirichlet L-function, we conjecture

$$\sum_{|d| \leqslant D}' L\left(\frac{1}{2}, \chi\right) \sim \frac{1}{2} \sum_{|d| \leqslant D}' \left(\log \frac{|d|}{\pi}\right) \prod_{p|d} \left(1 - \frac{1}{p}\right).$$

Similarly, for $p \equiv v(4)$ ($v = 1, 3$) with p a rational prime

$$\sum_{\substack{p \leqslant D \\ p \equiv v(4)}} L\left(\frac{1}{2}, \chi\right) \sim \frac{1}{2} \sum_{\substack{p \leqslant D \\ p \equiv v(4)}} \left(\left(\log \frac{p}{\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{v}{4}\right) + 4\gamma\right)$$

where γ is Euler's constant.

Let $S(x; v) = \sum_{p<x, p\equiv v(4)} L(\frac{1}{2}, \chi)$ and let $C(x; v)$ denote the conjectured value of $S(x, v)$ as given above. Harold Stark has provided the following numerical computations:

x	$S(x; 3)$	$C(x; 3)$	Ratio $= S(x; 3)/C(x; 3)$
151	34.7616	36.1663	.96116
163	34.8302	38.7365	.89916
383	99.5307	95.1260	1.04630
907	218.4371	225.3699	.96924
1483	367.6019	368.4617	.99767
1759	438.4471	442.9829	.98976
1907	480.9609	480.9418	1.00004

x	$S(x; 1)$	$C(x; 1)$	Ratio $= S(x; 1)/C(x; 1)$
313	34.797	23.674	1.470
701	90.259	70.681	1.277
1109	144.286	126.853	1.137
1601	215.379	188.781	1.141
1997	273.009	248.488	1.099

In the case when $v = 3$, there was a marked influence of prime conductors having low class numbers. The actual values come surprisingly close to the conjectured values and for the special case, $x = 1907$, one really could not hope for a better estimate.

3. LAVRIK'S METHOD

On using a method of Lavrik [6], we can write $L_1(k/2, \chi)$ as a rapidly converging series involving incomplete multiple gamma-functions.

Let $0 \leqslant \operatorname{Re} s \leqslant k$, $c \geqslant \sigma_0$. We have by (1.1)

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w_x A_x^{k-s+z} T_x(k-s+z)\, L_1(k-s+z,\chi)\frac{dz}{z}$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A_x^{s-z} T_x(s-z)\, L_1(s-z,\chi)\frac{dz}{z}$$
$$= A_x{}^s T_x(s)\, L_1(s,\chi) - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A_x^{s+z} T_x(s+z)\, L_1(s+z,\chi)\frac{dz}{z},$$

on shifting the line of integration from c to $-c$ and transforming z into $-z$. Hence

$$A_x{}^s T_x(s)\, L_1(s,\chi)$$
$$= A_x{}^s \sum_{n=1}^{\infty} a(n)\,\chi(n)\, n^{-s}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_x(s+z)\left(\frac{A_x}{n}\right)^z\frac{dz}{z}$$
$$+ w_x A_x^{k-s}\sum_{n=1}^{\infty} a(n)\,\chi(n)\, n^{-(k-s)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_x(k-s+z)\left(\frac{A_x}{n}\right)^z\frac{dz}{z}.$$

Choosing $s = k/2$ it follows that

$$L_1\left(\frac{k}{2},\chi\right) = \frac{1+w_x}{T_x(k/2)}\sum_{n=1}^{\infty} a(n)\,\chi(n)\, n^{-k/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_x\left(\frac{k}{2}+z\right)\left(\frac{A_x}{n}\right)^z\frac{dz}{z}. \tag{3.1}$$

We now have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_x\left(\frac{k}{2}+z\right)\left(\frac{A_x}{n}\right)^z\frac{dz}{z}$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\prod_{i=1}^{J}\int_0^{\infty} e^{-u}u^{k/2+z+\alpha_i-1}du\right)\left(\frac{A_x}{n}\right)^z\frac{dz}{z}$$
$$= \int_0^{\infty}\cdots\int_0^{\infty} e^{-(u_1+\cdots+u_J)}u_1^{k/2+\alpha_1-1}\cdots u_J^{k/2+\alpha_J-1}\,du_1\cdots du_J$$
$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(u_1\cdots u_J\frac{A_x}{n}\right)^z\frac{dz}{z}.$$

Since

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H^z\frac{dz}{z} = \begin{cases}0 & \text{if } 0 < H < 1\\ 1 & \text{if } H > 1\end{cases}$$

we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_x\left(\frac{k}{2}+z\right)\left(\frac{A_x}{n}\right)^z\frac{dz}{z}$$
$$= \underset{\substack{u_1\cdots u_J>n/A_x\\ u_i>0}}{\int\cdots\int} e^{-(u_1+\cdots+u_J)}u_1^{k/2+\alpha_1-1}\cdots u_J^{k/2+\alpha_J-1}\,du_1\cdots du_J. \tag{3.2}$$

4. The Incomplete Multiple Gamma-Function

For $x>0$ and for any complex numbers $\lambda_1,\ldots,\lambda_J$ ($J\geqslant 1$) we define the incomplete multiple gamma-function

$$G(x)=G(x;\lambda_1,\ldots,\lambda_J)$$
$$=\underset{\substack{u_1\cdots u_J>x^J\\ u_i>0}}{\int\cdots\int} e^{-(u_1+\cdots+u_J)}u_1^{\lambda_1-1}\cdots u_J^{\lambda_J-1}\,du_1\cdots du_J; \tag{4.1}$$

clearly if $\operatorname{Re}\lambda_i>0$ the function is also defined at $x=0$, and

$$G(0;\lambda_1,\ldots,\lambda_J)=\prod_{i=1}^{J}\Gamma(\lambda_i).$$

We remark that $G(x)$ can be estimated in an elementary manner. We require the following

Lemma. *Let $x>0$, $m>0$ and μ complex (m, μ fixed). For any $\epsilon>0$,*

$$\int_0^\infty e^{-x(t^m+mt^{-1})}t^\mu\,dt=\sqrt{\frac{2\pi}{m(m+1)\,x}}\;e^{-(m+1)x}(1+O(x^{-1/2+\epsilon}))\qquad(x\to\infty).$$

Proof. We split up the integral as $\int_{1-\delta}^{1+\delta}+\int_0^{1-\delta}+\int_{1+\delta}^{\infty}$, where $\delta=x^{-\beta}$ for a suitable constant β satisfying $\frac{1}{3}<\beta<\frac{1}{2}$. For any sufficiently large x, $e^{-x(t^m+mt^{-1})}t^{\operatorname{Re}\mu}$ is an increasing function of t in $0\leqslant t\leqslant 1-\delta$. Hence

$$\left|\int_0^{1-\delta} e^{-x(t^m+mt^{-1})}t^\mu\,dt\right|\leqslant\int_0^{1-\delta}e^{-x(t^m+mt^{-1})}t^{\operatorname{Re}\mu}\,dt$$
$$\leqslant(1-\delta)^{\operatorname{Re}\mu+1}\exp\left\{-x\left[(1-\delta)^m+\frac{m}{1-\delta}\right]\right\}$$
$$\ll\exp\left\{-(m+1)\,x\left[1+\frac{m\,\delta^2}{2}+O(\delta^3)\right]\right\}$$
$$=o(e^{-(m+1)x}x^{-\gamma}),$$

for any $\gamma > 0$. Similarly

$$\begin{aligned}\left|\int_{1+\delta}^{\infty} e^{-x(t^m+mt^{-1})} t^{\mu}\, dt\right| &= \left|\int_0^{1/(1+\delta)} e^{-x(t^{-m}+mt)} t^{-\mu-2}\, dt\right| \\ &\leqslant (1+\delta)^{\operatorname{Re}\mu+1} \exp\left\{-x\left[(1+\delta)^m + \frac{m}{1+\delta}\right]\right\} \\ &\ll \exp\left\{-(m+1)\,x\left[1+\frac{m\,\delta^2}{2}+O(\delta^3)\right]\right\} \\ &= o(e^{-(m+1)x}x^{-\gamma}),\end{aligned}$$

whence

$$\left(\int_0^{1-\delta} + \int_{1+\delta}^{\infty}\right) e^{-x(t^m+mt^{-1})} t^{\mu}\, dt = o(e^{-(m+1)x}x^{-\gamma}) \tag{4.2}$$

for any $\gamma > 0$. Next

$$\begin{aligned}\int_{1-\delta}^{1+\delta} e^{-x(t^m+mt^{-1})} t^{\mu}\, dt &= \int_{-\delta}^{\delta} \exp\left\{-x\left[(1+t)^m + \frac{m}{1+t}\right]\right\}(1+t)^{\mu}\, dt \\ &= (1+O(\delta))\, e^{-(m+1)x}\int_{-\delta}^{\delta} \exp\left\{-x\left[\frac{m(m+1)}{2}t^2 + O(|t|^3)\right]\right\} dt \\ &= (1+O(\delta))\, e^{-(m+1)x}\int_{-\delta}^{\delta} \exp\left[\frac{-m(m+1)}{2}xt^2\right](1+O(x\,\delta^3))\, dt \\ &= x^{-1/2}e^{-(m+1)x}\int_{-\delta x^{1/2}}^{\delta x^{1/2}} \exp\left[\frac{-m(m+1)}{2}t^2\right] dt\,(1+O(x^{1-3\beta})).\end{aligned}$$

Since

$$\begin{aligned}&\int_{-\delta x^{1/2}}^{\delta x^{1/2}} \exp\left[-\frac{m(m+1)}{2}t^2\right] dt \\ &\quad= \int_{-\infty}^{\infty} \exp\left[-\frac{m(m+1)}{2}t^2\right] dt - 2\int_{\delta x^{1/2}}^{\infty} \exp\left[-\frac{m(m+1)}{2}t^2\right] dt \\ &\quad= \sqrt{\frac{2\pi}{m(m+1)}}\left[1 + O\left(x^{-1/2}\delta^{-1}\exp\left[-\frac{m(m+1)}{2}x\,\delta^2\right]\right)\right] \\ &\quad= \sqrt{\frac{2\pi}{m(m+1)}}\,(1+o(x^{-\gamma}))\end{aligned}$$

for any $\gamma > 0$, we have

$$\int_{1-\delta}^{1+\delta} e^{-x(t^m+mt^{-1})} t^{\mu}\, dt = \sqrt{\frac{2\pi}{m(m+1)\,x}}\, e^{-(m+1)x}(1+O(x^{1-3\beta})).$$

Hence the choice $\beta = \frac{1}{2} - \epsilon/3$ yields

$$\int_{1-\delta}^{1+\delta} e^{-x(t^m+mt^{-1})} t^{\mu}\, dt = \sqrt{\frac{2\pi}{m(m+1)\,x}}\, e^{-(m+1)x}(1 + O(x^{-1/2+\epsilon})),$$

which, together with (4.2), proves the lemma.

THEOREM (2). *For fixed complex $\lambda_1, \ldots, \lambda_J$ and for any $\epsilon > 0$,*

$$G(x; \lambda_1, \ldots, \lambda_J) = \sqrt{\frac{(2\pi)^{J-1}}{J}}\, e^{-Jx} x^{\sum_1^J \lambda_i - (J+1)/2}(1 + O(x^{-1/2+\epsilon})) \qquad (x \to \infty).$$

Proof. By induction on J. Since

$$\int_x^{\infty} e^{-u} u^{\lambda-1}\, du = e^{-x} x^{\lambda-1} + (\lambda - 1) \int_x^{\infty} e^{-u} u^{\lambda-2}\, du = e^{-x} x^{\lambda-1}(1 + O(x^{-1})),$$

the theorem holds for $J = 1$. Hence we let $J \geqslant 2$ and assume the theorem for $J - 1$ in place of J. Then

$$\begin{aligned}
G(x) &= \int_0^{\infty} e^{-u_J} u_J^{\lambda_J - 1}\, du_J \underset{\substack{u_1 \cdots u_{J-1} > x^J/u_J \\ u_i > 0}}{\int \cdots \int} e^{-(u_1 + \cdots + u_{J-1})} u_1^{\lambda_1 - 1} \cdots u_{J-1}^{\lambda_{J-1}-1}\, du_1 \cdots du_{J-1} \\
&= \int_0^{\infty} e^{-u} u^{\lambda_J - 1} G\left(\left(\frac{x^J}{u}\right)^{1/(J-1)}; \lambda_1, \ldots, \lambda_{J-1}\right) du \\
&= (J-1)\, x^{\lambda_J} \int_0^{\infty} e^{-xt^{J-1}} t^{(J-1)\lambda_J - 1} G\left(\frac{x}{t}; \lambda_1, \ldots, \lambda_{J-1}\right) dt \\
&= \sqrt{(J-1)(2\pi)^{J-2}}\, x^{\sum_1^J \lambda_i - J/2} \int_0^{\infty} \exp\left\{-x\left(t^{J-1} + \frac{J-1}{t}\right)\right\} \\
&\quad \times t^{\mu} \left\{1 + O\left(\left(\frac{x}{t}\right)^{-1/2+\epsilon}\right)\right\} dt
\end{aligned}$$

for some constant μ, namely $\mu = J/2 - 1 - \sum_1^{J-1} \lambda_i + (J-1)\lambda_J$. By the lemma,

$$\begin{aligned}
G(x) &= \sqrt{(J-1)(2\pi)^{J-2}}\, x^{\sum_1^J \lambda_i - J/2} \sqrt{\frac{2\pi}{J(J-1)\,x}}\, e^{-Jx}(1 + O(x^{-1/2+\epsilon})) \\
&= \sqrt{\frac{(2\pi)^{J-1}}{J}}\, e^{-Jx} x^{\sum_1^J \lambda_i - (J+1)/2}(1 + O(x^{-1/2+\epsilon})).
\end{aligned}$$

5. Proof of Theorem 1

In order to simplify the subsequent notation, we put

$$G_n(t) = G\left(\left(\frac{n}{f(|t|)}\right)^{1/J}; \frac{k}{2} + \alpha_1, \ldots, \frac{k}{2} + \alpha_J\right),$$

where f is the function defined by (1.3). By (3.1), (3.2) and (4.1) we obtain

$$\begin{aligned} L_1\left(\frac{k}{2}, \chi\right) &= \frac{1 + w_\chi}{T_\chi(k/2)} \sum_{n=1}^{\infty} a(n)\, G_n(d)\, \chi(n)\, n^{-k/2} \\ &= \frac{1 + w_\chi}{T_\chi(k/2)} \left(\sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} a(m^2)\, G_{m^2}(d)\, m^{-k} + \sum_{\substack{n=1 \\ n \neq m^2}}^{\infty} a(n)\, G_n(d)\, \chi(n)\, n^{-k/2} \right) \\ &= \Sigma_1 + \Sigma_2 . \end{aligned}$$

Letting ψ denote the principal character mod $|d|$, it follows that for sufficiently large c

$$\begin{aligned} \Sigma_1 &= \frac{1 + w_\chi}{T_\chi(k/2)} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} a(m^2)\, m^{-k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} T_\chi\left(\frac{k}{2} + z\right) \left(\frac{f(|d|)}{m^2}\right)^z \frac{dz}{z} \\ &= \frac{1 + w_\chi}{T_\chi(k/2)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} T_\chi\left(\frac{k}{2} + z\right) L_2(k + 2z, \psi)\, f(|d|)^z \frac{dz}{z} . \end{aligned}$$

Summing over $|d| \leqslant D$ gives the main term in Theorem (1).

To complete the proof of Theorem (1) it only remains to show that

$$\sum_{|d| \leqslant D}{}' \sum_{2} \ll D^{1/2}(\log D) \left(1 + \sum_{n \leqslant F} |a(n)|\, n^{-k/2+3/16+\epsilon}\right). \tag{5.1}$$

6. Estimation of the Average of Σ_2

In order to prove (5.1) it is necessary to consider separately positive and negative fundamental discriminants d. We work out the case $d > 0$, since the arguments are the same for $d < 0$.

By (1.2) we can write

$$T_\chi(s) = T(s) = \prod_{i=1}^{J} \Gamma(s + \alpha_i) \qquad \text{for} \quad d > 0,$$

and therefore

$$\sum_{0<d\leqslant D}' \sum_{2} = \frac{1}{T(k/2)} \sum_{\substack{n=1\\ n\neq m^2}}^{\infty} a(n)\, n^{-k/2} \sum_{0<d\leqslant D}' (1 + w_x)\, \chi(n)\, G_n(d)$$

$$= \frac{1}{T(k/2)} \sum_{\substack{n=1\\ n\neq m^2}}^{\infty} a(n)\, n^{-k/2} \cdot S_n \,. \tag{6.1}$$

By partial summation,

$$S_n = \sum_{0<d\leqslant D}' \left(1 + w\epsilon(d)\left(\frac{d}{R}\right)\right)\left(\frac{d}{n}\right) G_n(D)$$

$$- \int_1^D \left\{ \sum_{0<d\leqslant t}' \left(1 + w\epsilon(d)\left(\frac{d}{R}\right)\right)\left(\frac{d}{n}\right)\right\} dG_n(t);$$

since $G_n(t)$ is an increasing function of t, we obtain

$$| S_n | \leqslant 2 \max_{x\leqslant D} \left| \sum_{0<d\leqslant x}' \left(1 + w\epsilon(d)\left(\frac{d}{R}\right)\right)\left(\frac{d}{n}\right)\right| G_n(D). \tag{6.2}$$

Now, let $n = 2^h n_0 u v^2$, $n_0 u v$ odd, $n_0 u$ squarefree, and $(n_0\,, NR) = 1$. Then

$$\left(\frac{d}{n}\right) = \left(\frac{d}{2}\right)^h \left(\frac{d}{n_0}\right)\left(\frac{d}{u}\right)\left(\frac{d}{v}\right)^2$$

where (d/n_0) can be considered as a Jacobi symbol, and is therefore a primitive character mod n_0 (see [5]). In order to estimate

$$\sum_{0<d\leqslant x}' \left(1 + w\epsilon(d)\left(\frac{d}{R}\right)\right)\left(\frac{d}{n}\right)$$

it is easy to see that it is enough to estimate the sum

$$\tilde{S}_{a,b}(x) = \sum_{\substack{0<d\leqslant x\\ d\equiv a(\text{mod } b)\\ d \text{ squarefree}}} \left(\frac{d}{n_0}\right),$$

where a and b are bounded and depend at most on N and R, and where $(b, n) = 1$. We have

$$\tilde{S}_{a,b}(x) = \sum_{\substack{0<d\leqslant x\\ d\equiv a(b)}} \left(\frac{d}{n_0}\right) \sum_{\delta^2 | d} \mu(\delta) = \sum_{\delta\leqslant x^{1/2}} \mu(\delta) \sum_{\substack{d\leqslant x\\ d\equiv a(b)\\ d\equiv 0(\delta^2)}} \left(\frac{d}{n_0}\right)$$

$$= \sum_{\delta<x^{1/2}} \mu(\delta) \sum_{\substack{d_0\leqslant x/\delta^2\\ d_0\delta^2\equiv a(b)}} \left(\frac{d_0}{n_0}\right)\left(\frac{\delta}{n_0}\right)^2 = \sum_{\substack{\delta<x^{1/2}\\ (\delta, n_0)=1}} \mu(\delta) \sum_{\substack{d_0\leqslant x/\delta^2\\ d_0\equiv a_0(b_0)}} \left(\frac{d_0}{n_0}\right).$$

We let $d_0 = a_0 + \rho b_0$ in the inner sum, and denote by b_0^{-1} the inverse of $b_0 \bmod n_0$. Then

$$\tilde{S}_{a,b}(x) = \sum_{\substack{\delta \leqslant x^{1/2} \\ (\delta, n_0)=1}} \mu(\delta) \left(\frac{b_0}{n_0}\right) \sum_{\rho} \left(\frac{a_0 b_0^{-1} + \rho}{n_0}\right),$$

where ρ runs over the range corresponding to $d_0 \leqslant x/\delta^2$. By (2.3) we obtain

$$\sum_{\rho} \left(\frac{a_0 b_0^{-1} + \rho}{n_0}\right) \ll \left(\frac{x}{\delta^2}\right)^{1/2} n_0^{3/16+\epsilon}.$$

Hence

$$\tilde{S}_{a,b}(x) \ll \sum_{\delta \leqslant x^{1/2}} \frac{x^{1/2}}{\delta} n_0^{3/16+\epsilon} \ll x^{1/2}(\log x)\, n_0^{3/16+\epsilon}.$$

By (6.2),

$$\begin{aligned} S_n &\ll \max_{x \leqslant D} \sum_{a,b} |\tilde{S}_{a,b}(x)| \cdot G_n(D) \\ &\ll D^{1/2}(\log D)\, n^{3/16+\epsilon} G_n(D). \end{aligned}$$

From this and (6.1) it follows that

$$\begin{aligned} \sum_{0<d\leqslant D}{}' \sum_{2} &\ll D^{1/2}(\log D) \sum_{n=1}^{\infty} |a(n)|\, n^{-k/2+3/16+\epsilon} G_n(D) \\ &\ll D^{1/2}(\log D) \Big(\sum_{n<F} |a(n)|\, n^{-k/2+3/16+\epsilon} + \sum_{n>F} |a(n)|\, n^{-k/2+3/16+\epsilon} G_n(D) \Big), \end{aligned}$$

where F is as in the statement of Theorem (1). Since $|a(n)|$ has at most polynomial growth, the sum over $n > F$ above is easily shown to be $O(1)$, on using the estimate

$$G_n(D) \ll \exp\left[-J \left(\frac{n}{f(D)}\right)^{1/J}\right] \left(\frac{n}{f(D)}\right)^{k/2+(1/J)\sum_1^J \alpha_i - (J+1)/2J}$$

given by Theorem (2). This completes the proof of (5.1) and therefore the proof of Theorem (1).

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